Explore PASSIVE rotations and EULER rates

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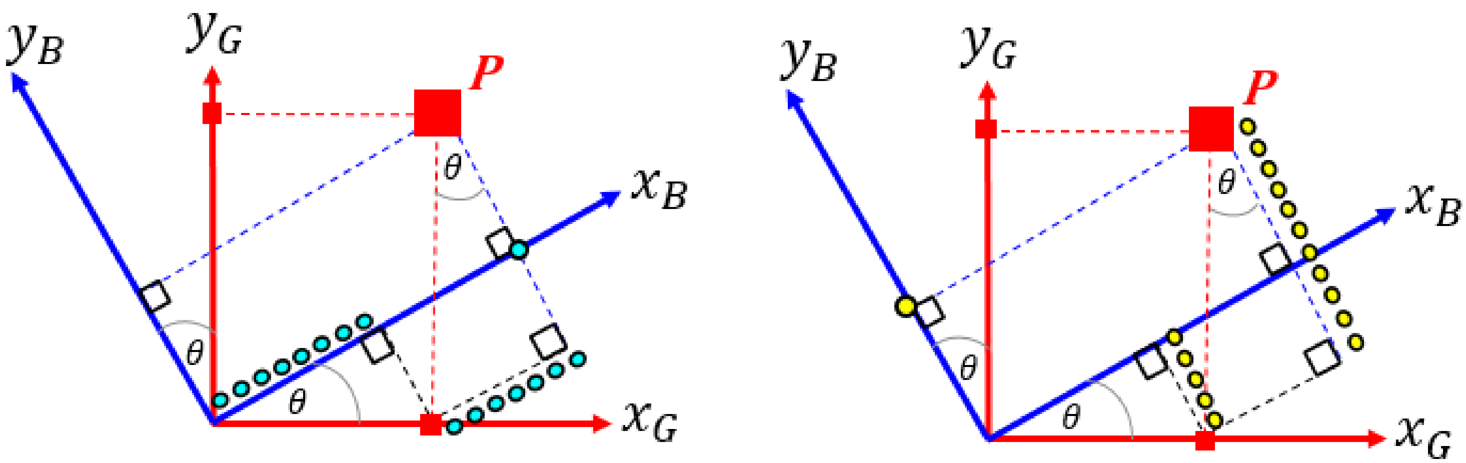
Introduction:

A Passive rotation matrix, converts the co-ordinates of a point expressed in a fixed **G-frame**, into the co-ordinates of the same point expressed in the new **B-frame**.

An example of this concept is shown below



$$\begin{bmatrix} x_P \\ y_P \end{bmatrix}_B = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} * \begin{bmatrix} x_P \\ y_P \end{bmatrix}_G$$

An example of 3 successive PASSIVE rotations

Say we start with a G-frame. We're going to apply 3 LOCAL axes rotations which will result in a newly orientated frame called the B-frame.

Assume that we apply these 3 successive rotations in the following order:

1. R1Z occurs 1st about the LOCAL **Z** body axis (ϕ), aka **YAW**
2. R2Y occurs 2nd about the LOCAL **Y** body axis (θ), aka **PITCH**
3. R3X occurs 3rd about the LOCAL **X** body axis (ψ), aka **ROLL**

We can express a vector defined in the G axis into it's corresponding description in the B axis, using a **PASSIVE** rotation matrix, ie:

vB = R3X(ψ_x) * R2Y(θ_y) * R1Z(ϕ_z) * vG

OR, in a more compact form as:

vB = bRg * vG

Create a passive rotation object

```
syms phi theta psi
OBJ_P = bh_rot_passive_G2B_CLS({'D1Z', 'D2Y', 'D3X'}, [phi, theta, psi], 'SYM')
```

```
OBJ_P = 
  bh_rot_passive_G2B_CLS with properties:

        ang_units: SYM
    num_rotations: 3
          dir_1st: D1Z
          dir_2nd: D2Y
          dir_3rd: D3X
          ang_1st: [1x1 sym]
          ang_2nd: [1x1 sym]
          ang_3rd: [1x1 sym]
```

Here are the PASSIVE rotation matrices

```
R1 = OBJ_P.get_R1
```

R1 =

$$\begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
R2 = OBJ_P.get_R2
```

R2 =

$$\begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

```
R3 = OBJ_P.get_R3
```

R3 =

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi) & \sin(\psi) \\ 0 & -\sin(\psi) & \cos(\psi) \end{pmatrix}$$

Calculate the Direction Cosine Matrix ${}^{B}R_{G}$

Recall we earlier said: ${}^{B}v = {}^{B}R_{G} * {}^{G}v$

```
bRg = R3 * R2 * R1
```

```
bRg =
```

$$\begin{pmatrix} \cos(\varphi)\cos(\theta) & \cos(\theta)\sin(\varphi) & -\sin(\theta) \\ \cos(\varphi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\varphi) & \cos(\varphi)\cos(\psi) + \sin(\varphi)\sin(\psi)\sin(\theta) & \cos(\theta)\sin(\psi) \\ \sin(\varphi)\sin(\psi) + \cos(\varphi)\cos(\psi)\sin(\theta) & \cos(\psi)\sin(\varphi)\sin(\theta) - \cos(\varphi)\sin(\psi) & \cos(\psi)\cos(\theta) \end{pmatrix}$$

And it's nice to know I can automatically convert this into a MATLAB function.

NOTE: we're specifying the order of the input variables for the function that gets generated.

```
matlabFunction(bRg,'File','bh_autogen_bRg','Optimize',false, 'Vars', {'phi','theta', 'psi'});

% look at the first 6 lines of this autogenerated file
dbtype('bh_autogen_bRg', '1:6')
```

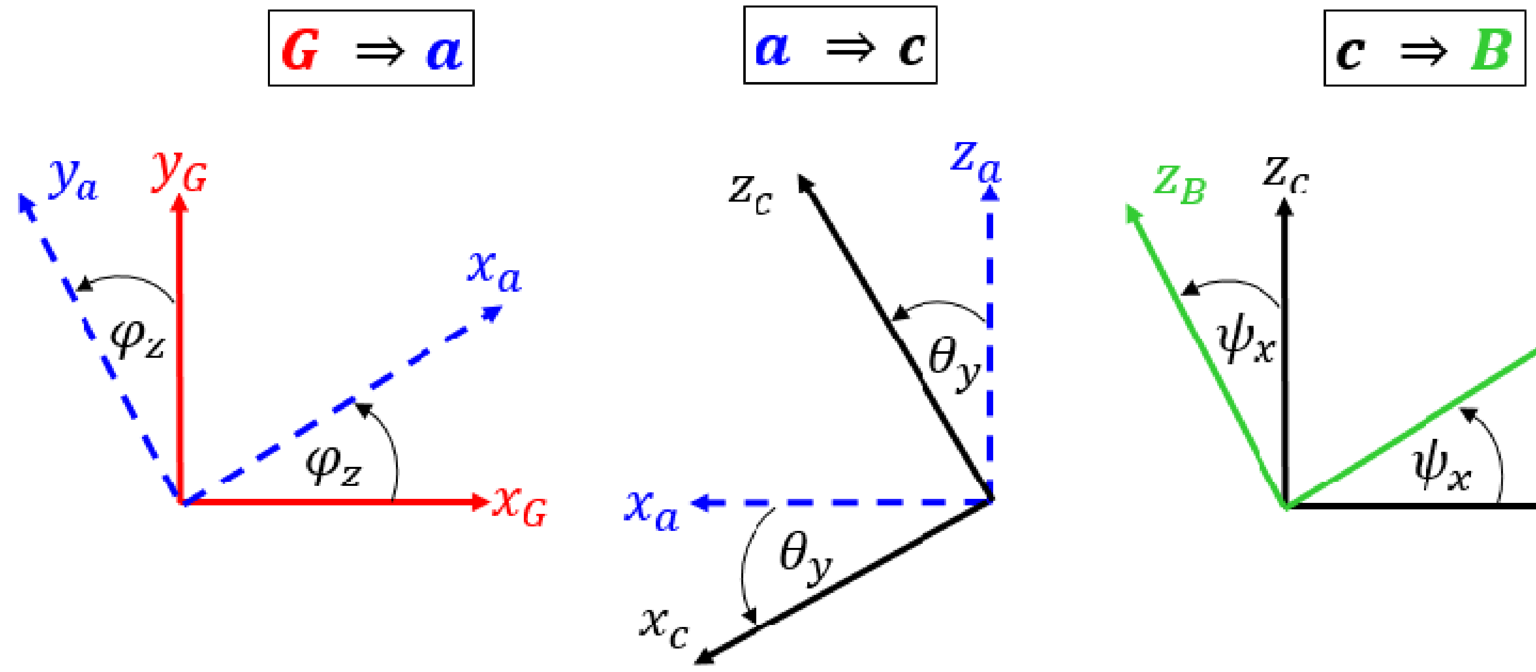
```
1     function bRg = bh_autogen_bRg(phi,theta,psi)
2     %BH_AUTOGEN_BRG
3     %    BRG = BH_AUTOGEN_BRG(PHI,THETA,PSI)
4
5     %    This function was generated by the Symbolic Math Toolbox version 7.0.
6     %    07-Mar-2016 16:47:46
```

Explore EULER rates

As we apply these local frame rotations, we can represent the angular rates of the rotating rames in the LOCAL frame co-ordinates. These local frame co-ordinates can then be converted into co-ordinates expressed in the final B frame.

For example, during each of the local axes rotations we can think of there being a START frame and an END frame:

```
START END Angular rate vector

frame frame associated with rotation

-------------------------------------------------------------------

R1Z(phi) G_frame a_frame [0 0 phi_dot]_G

R2Y(theta) a_frame c_frame [0 theta_dot 0 ]_a

R3X(psi) c_frame B_frame [psi_dot 0 0 ]_c
```



We can express each of the local frame angular velocities into their corresponding components in the B frame - and we'll use PASSIVE rotation matrices to do this:

```
syms phi_dot  theta_dot  psi_dot

aRg = R1;
cRa = R2;
bRc = R3;

wb_part_1 = bRc * cRa * aRg * [0;0;phi_dot]   % convert local G into B
```

wb_part_1 =

$$\begin{pmatrix} -\varphi_{\mathrm{dot}}\sin(\theta) \\ \varphi_{\mathrm{dot}}\cos(\theta)\sin(\psi) \\ \varphi_{\mathrm{dot}}\cos(\psi)\cos(\theta) \end{pmatrix}$$

```
wb_part_2 = bRc * cRa *       [0;theta_dot;0] % convert local a into B
```

wb_part_2 =

$$\begin{pmatrix} 0 \\ \theta_{\mathrm{dot}}\cos(\psi) \\ -\theta_{\mathrm{dot}}\sin(\psi) \end{pmatrix}$$

```
wb_part_3 = bRc *             [psi_dot;0;0]   % convert local c into B
```

wb_part_3 =

$$\begin{pmatrix} \psi_{\mathrm{dot}} \\ 0 \\ 0 \end{pmatrix}$$

The total angular velocity expressed in the BODY B frame is therefore

We can now construct the total angular velocity vector expressed in components of the final B frame.

$$ {}^{B}_{G}\omega_b \equiv \omega_b = f(\phi_{dot}, \theta_{dot}, \psi_{dot}) $$

```
wb = wb_part_1 + wb_part_2 + wb_part_3
```

```
wb =
```

$$\begin{pmatrix} \psi_{\mathrm{dot}} - \varphi_{\mathrm{dot}} \sin(\theta) \\ \theta_{\mathrm{dot}} \cos(\psi) + \varphi_{\mathrm{dot}} \cos(\theta) \sin(\psi) \\ \varphi_{\mathrm{dot}} \cos(\psi) \cos(\theta) - \theta_{\mathrm{dot}} \sin(\psi) \end{pmatrix}$$

We can write the angular velocity vector $\boldsymbol{\omega_b}$ as a MATRIX equation

Let's say that: $\omega_b = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

We can write a matrix equation of the form **A.x = b** that describes the relationship between the body rates ω_b and the Euler rates:

$$ A * x = b $$

$$ A * \begin{pmatrix} \phi_{dot} \\ \theta_{dot} \\ \psi_{dot} \end{pmatrix} = \omega_b \equiv \begin{pmatrix} p \\ q \\ r \end{pmatrix} $$

```
syms p q r

    x = [phi_dot, theta_dot, psi_dot].'
```

```
x =
```

$$\begin{pmatrix} \varphi_{\mathrm{dot}} \\ \theta_{\mathrm{dot}} \\ \psi_{\mathrm{dot}} \end{pmatrix}$$

```
[A,b] = equationsToMatrix(  wb(1)==p, ...
                            wb(2)==q, ...
                            wb(3)==r, ...
                            x)
```

A =

$$\begin{pmatrix} -\sin(\theta) & 0 & 1 \\ \cos(\theta)\sin(\psi) & \cos(\psi) & 0 \\ \cos(\psi)\cos(\theta) & -\sin(\psi) & 0 \end{pmatrix}$$

b =

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

ATTENTION: The SINGULARITY between BODY rates and EULER rates

From the Matrix equation computed above there is actually an angle that causes the determinant of **A** to be ZERO, and hence prevents us from solving for the Euler rates iff we know the body rates ω_b.

The angle that causes this problem is the rotation about the local Y axis, ie: the angle **phi**. Specifically it is when **phi = 90 degrees**.

We can see this by first computing the determinant

```
det_A = simplify( det(A) )
```

det_A $= -\cos(\theta)$

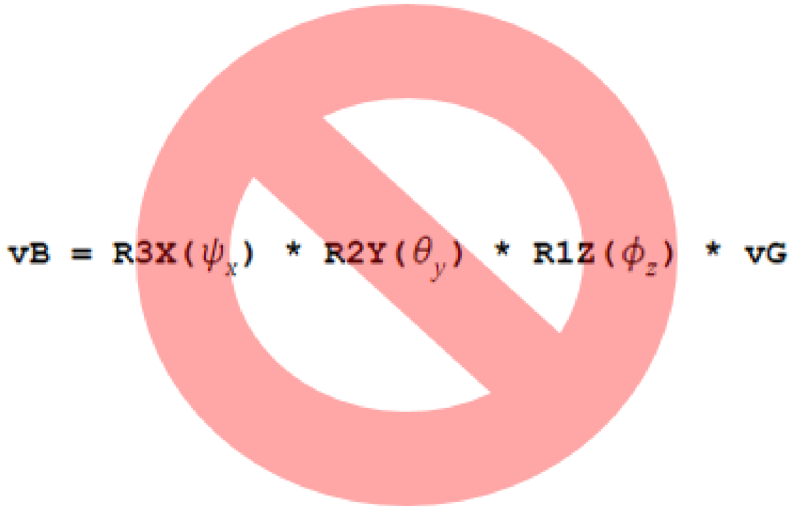
And then solving for its roots.

```
solve( det_A ==0 )
```

ans $= \frac{\pi}{2}$

So this tells us that as soon as our vehicle has a pitch angle of 90 degrees, that our chosen Euler angle sequnce simply canNOT be used to convert body rates ω_b into Euler rates.

If you think your vehicle will pitch by 90 degrees, then you'll need to consider an alternate form of describing your vehicle's pose (eg: quaternions, or integrating directly the DCM)



Let's compute Euler rates from our body rates ω_b

Assuming our vehicle does NOT have a pitch angle of 90 degrees, then we can use the results of the previous section to calculate the Euler rates from our body rates ω_b.

$$Euler_{rates} \equiv \begin{pmatrix} \phi_{dot} \\ \theta_{dot} \\ \psi_{dot} \end{pmatrix} = A^{-1} * \omega_b$$

```
euler_rates = inv(A) * [p; q; r];
euler_rates = simplify(euler_rates)
```

```
euler_rates =
```

$$\begin{pmatrix} \frac{r\cos(\psi) + q\sin(\psi)}{\cos(\theta)} \\ q\cos(\psi) - r\sin(\psi) \\ \frac{p\cos(\theta) + r\cos(\psi)\sin(\theta) + q\sin(\psi)\sin(\theta)}{\cos(\theta)} \end{pmatrix}$$

We can write the Euler rates vector as a MATRIX equation

Similarly to what we did earlier we can write a matrix equation that describes the relationship between the body rates ω_b and the Euler rates:

$$K * \omega_b = Euler_{rates}$$

$$K * \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \phi_{dot} \\ \theta_{dot} \\ \psi_{dot} \end{pmatrix}$$

$$K * x = b$$

```
x = [p,q,r].'
```

x =

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

```
[K,b] = equationsToMatrix(  euler_rates(1)==phi_dot,   ...
                            euler_rates(2)==theta_dot, ...
                            euler_rates(3)==psi_dot,   ...
                            x)
```

K =

$$\begin{pmatrix} 0 & \frac{\sin(\psi)}{\cos(\theta)} & \frac{\cos(\psi)}{\cos(\theta)} \\ 0 & \cos(\psi) & -\sin(\psi) \\ 1 & \frac{\sin(\psi)\sin(\theta)}{\cos(\theta)} & \frac{\cos(\psi)\sin(\theta)}{\cos(\theta)} \end{pmatrix}$$

b =

$$\begin{pmatrix} \varphi_{\mathrm{dot}} \\ \theta_{\mathrm{dot}} \\ \psi_{\mathrm{dot}} \end{pmatrix}$$