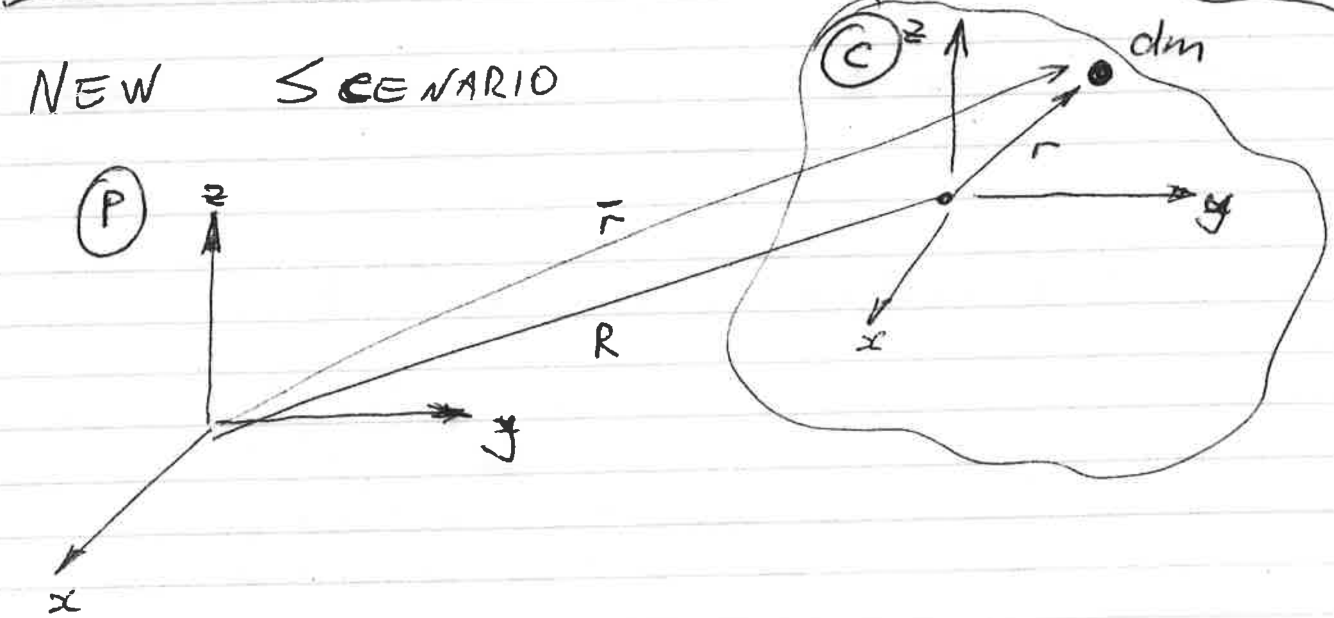
with the last result showing the presence of the "PARALLEL AXIS THEOREM for the INERTIA matrix".

NEW SCENARIO



Let:-

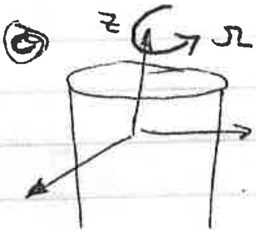
1.) FRAME (C) is attached to the centre of mass of the body. .... and is "ALMOST" body fixed.

2.) The body has an angular velocity relative to the (C) frame, such that

$$\bullet\ v_{dm} = v_{oc} + \omega \times r + \Omega \times r$$
$$= v_{oc} + (\omega + \Omega) \times r$$

where:- ω ≡ angular velocity of (C) FRAME
Ω ≡ angular velocity of BODY relative to the (C) FRAME.

3.) We'll assume to that the INERTIA of the body according to FRAME (C) does NOT change .... even though the body is rotating relative to FRAME (C). An example of this situation is



ok: as before we can write:-

$$v = v_{oc} + (\omega + \Omega) \times r$$

$\therefore$ the angular momentum relative to the P-frame is:-

$${}^{P}_{o}L = \int \bar{r} \times v \, dm$$

$${}^{P}_{o}L = \int (R + r) \times (v_o + (\omega + \Omega) \times r) \, dm$$

$$\begin{aligned} {}^{P}_{o}L = \; & \int R \times v_o dm \; + && \text{— 1} \\ & \int R \times (\omega + \Omega) \times r \, dm \; + && \text{— 2} \\ & \int r \times v_o dm \; + && \text{— 3} \\ & \int r \times (\omega + \Omega) \times r \, dm && \text{— 4} \end{aligned}$$

$\therefore$ Considering each of the 4 parts:-

PART_1 $= \int R \times v_o dm = R \times v_o \int dm = M R \times v_o$

PART_2 $= \int R \times (\omega + \Omega) \times r \, dm$
$= R \times (\omega + \Omega) \times \int r \, dm$, but $\int r \, dm = 0$ since Ⓒ is at c.o.m.
$= 0$

PART_3 $= \int r \times v_o dm$
$= -\int v_o \times r \, dm$
$= -v_o \times \int r \, dm$, since $\int r \, dm = 0$
$= 0$

PART_4 $= \int r \times (\omega + \Omega) \times r \, dm$
$= \int r \times (\omega \times r + \Omega \times r) \, dm$

$= \int r \times \omega \times r \, dm + \int r \times \Omega \times r \, dm$

$$\therefore \; {}^{P}_{o}L = M . R \times v_{oc} + \int r \times \omega \times r \, dm + \int r \times \Omega \times r \, dm$$

$$\boxed{{}^{P}_{o}L = M . R \times v_{oc} + [{}^{c}I] . \omega + [{}^{c}I] . \Omega}$$ — BIG RESULT

$$\therefore \; {}^{P}_{O}L = M\,R \times v_{oc} + [{}^{c}I].(\omega + \Omega) \quad -(2)$$

as before, we have :- $v_o = \omega \times R$

$$\therefore \; {}^{P}_{O}L = M . R \times \omega \times R + {}^{c}I(\omega + \Omega)$$

$$= M . \begin{bmatrix} Y^2+Z^2 & -XY & -XZ \\ -XY & X^2+Z^2 & -YZ \\ -XZ & -YZ & X^2+Y^2 \end{bmatrix} . \omega + {}^{c}I(\omega+\Omega)$$

$$= M.[D].\omega + [{}^{c}I]\omega + [{}^{c}I]\Omega$$

$$\boxed{{}^{P}_{O}L = (M.[D] + {}^{c}I)\omega + [{}^{c}I]\Omega} \quad -(3)$$

As before we can apply the DERIVATIVE TRANSFORMATION formulae to compute the rate of change of the angular momentum vector, ie:

$${}^{P}M = \frac{{}^{G}d}{dt}\left({}^{P}L\right) = \frac{{}^{P}d}{dt}\left({}^{P}L\right) + \omega \times {}^{P}L$$

$\cancel{\therefore \; {}^{P}M = (M[D] + {}^{c}I)\dot{\omega} + \omega \times}$

$$\therefore \; {}^{P}M = (M[D] + {}^{c}I)\dot{\omega} + [{}^{c}I]\dot{\Omega} + \omega \times \Big((MD + {}^{c}I)\omega + {}^{c}I\Omega \Big)$$

$$\boxed{{}^{P}M = (M[D] + {}^{c}I)\dot{\omega} + \omega \times \{(M[D] + {}^{c}I).\omega\} + {}^{c}I\dot{\Omega} + \omega \times \{{}^{c}I.\Omega\}}$$

PART (1)

PART (2)

So PART (1) are just the standard EULER components. And PART (2) are the new terms that arise because of the <u>extra</u> angular momentum of the system.

Let : • $\,^{P}I = M[D] + \,^{c}I$

$$\therefore \,^{P}M = \,^{P}I\dot{\omega} + \omega \times (\,^{P}I\omega) + I_c\dot{\Omega} + \omega \times (\,^{c}I\Omega)$$

$$\therefore \,^{P}M - \{ I_c\dot{\Omega} + \omega \times (\,^{c}I\Omega) \} = \,^{P}I\dot{\omega} + \omega \times (\,^{P}I\omega) \quad (4)$$

Let's look at 2 special cases of equation (4):-

CASE 1: $I_c = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$, $\Omega = \begin{pmatrix} 0 \\ 0 \\ \Omega_z \end{pmatrix}$

$$\therefore I_c . \dot{\Omega} = \begin{pmatrix} 0 \\ 0 \\ I_z . \dot{\Omega}_z \end{pmatrix}$$

$$\therefore \omega \times (\,^{c}I\Omega) = \begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ 0 & 0 & I_z\Omega_z \end{vmatrix} = \begin{pmatrix} \omega_y I_z \Omega_z \\ -\omega_x I_z \Omega_z \\ 0 \end{pmatrix}$$

$\therefore$ (4) becomes:- $$\,^{P}M - \begin{pmatrix} \omega_y I_z \Omega_z \\ -\omega_x I_z \Omega_z \\ I_z \dot{\Omega}_z \end{pmatrix} = \,^{P}I\dot{\omega} + \omega \times (\,^{P}I\omega)$$

Case 2: let $\Omega = \begin{pmatrix} \Omega_x \\ 0 \\ 0 \end{pmatrix}$ $\therefore I_c\dot{\Omega} = \begin{pmatrix} I_x\dot{\Omega}_x \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \omega \times (\,^{c}I\Omega) = \begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ I_x\Omega_x & 0 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ \omega_z I_x \Omega_x \\ -\omega_y I_x \Omega_x \end{pmatrix}$$

$\therefore$ (4) becomes $$\,^{P}M - \begin{pmatrix} I_x\dot{\Omega}_x \\ \omega_z I_x \Omega_x \\ -\omega_y I_x \Omega_x \end{pmatrix} = I_P\dot{\omega} + \omega \times (\,^{P}I\omega)$$