Explore **PASSIVE** rotations and **EULER rates**

In this tutorial we're going to look at how the EULER rates of a rigid body can be determined from the BODY rates of the rigid body. We'll see that there are certain angular poses that result in a matrix singularity which in turn prevents us from transforming from body rates to Euler rates. This tutorial demonstrates how PASSIVE rotation matrices can be applied.

Why are we doing this ?

• Before we can create a 6-DOF model of a vehicle (eg: a quadcopter), we need to get comfortable with certain concepts. Concepts such as PASSIVE rotation matrices.

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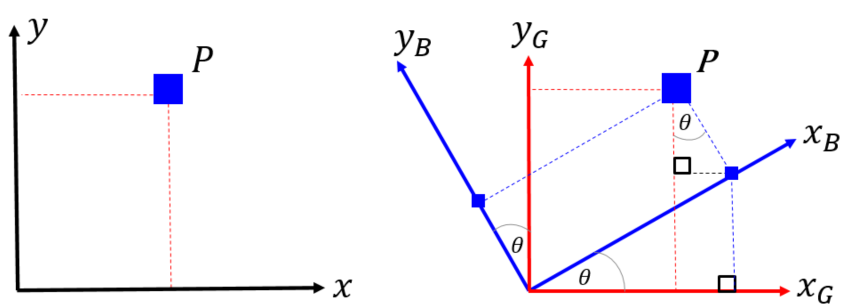
Review the concept of PASSIVE rotation matrices :

Consider the following scenario:

- We have a data point P.
- We have a fixed frame called the **G-fame.**
- We know the *(x,y)* co-ordinates of the point P in this **G-fame** and refer to this as ${}^{G}P$.
- We then rotate the **B-frame** relative to the fixed **G-frame**.

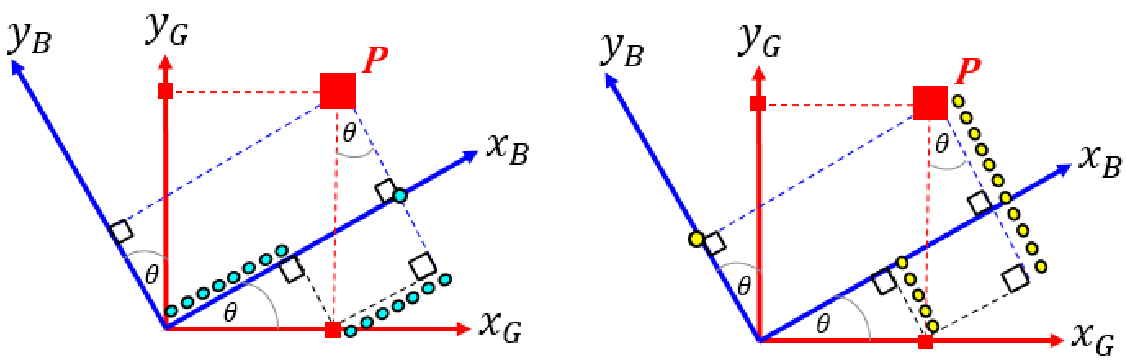
We now want to know what the co-ordinate of the point P is relative to this new **B-frame,** ie: what is ${}^{B}P$? This scenario is shown in the figure below:



$$\begin{bmatrix} x_P \\ y_P \end{bmatrix}_B = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} * \begin{bmatrix} x_P \\ y_P \end{bmatrix}_G$$

$${}^{B}\mathrm{P} = {}^{B}\mathrm{R}_{G} * {}^{G}\mathrm{P}$$

A **PASSIVE** rotation matrix, converts the co-ordinates of a point expressed in a fixed **G-frame**, into the co-ordinates of the same point expressed in the new **B-frame**.



$$\begin{bmatrix} x_P \\ y_P \end{bmatrix}_B = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} * \begin{bmatrix} x_P \\ y_P \end{bmatrix}_G$$

An example of 3 successive PASSIVE rotations

Say we have a fixed G-frame. We start by having our B-frame co-incident with G, and then we start to rotate the B-frame. Specifically, we're going to apply 3 LOCAL axes rotations which will result in a newly orientated B-frame. Assume that we apply these 3 successive rotations in the following order:

1. R1Z occurs 1st about the LOCAL **Z** body axis (ϕ), aka **YAW**
2. R2Y occurs 2nd about the LOCAL **Y** body axis (θ), aka **PITCH**
3. R3X occurs 3rd about the LOCAL **X** body axis (ψ), aka **ROLL**

We can express a vector defined in the G axis to it's corresponding description in the B axis, using a sequence of **PASSIVE** rotation matrices, ie:

$$^{B}\mathbf{v} = \mathbf{R3X}(\psi_x) \times \mathbf{R2Y}(\theta_y) \times \mathbf{R1Z}(\phi_z) \times {}^{G}\mathbf{v}$$

OR, in a more compact form as:

$$^{B}\mathbf{v} = {}^{B}\mathbf{R}_{G} \times {}^{G}\mathbf{v}$$

Create a passive rotation object

```
syms phi theta psi
OBJ_P = bh_rot_passive_G2B_CLS({'D1Z', 'D2Y', 'D3X'}, [phi, theta, psi], 'SYM')
```

```
OBJ_P =
  bh_rot_passive_G2B_CLS with properties:

        ang_units: SYM
    num_rotations: 3
          dir_1st: D1Z
          dir_2nd: D2Y
          dir_3rd: D3X
          ang_1st: [1x1 sym]
          ang_2nd: [1x1 sym]
          ang_3rd: [1x1 sym]
```

Here are the PASSIVE rotation matrices

```
R1 = OBJ_P.get_R1
```

R1 =

$$\begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
R2 = OBJ_P.get_R2
```

R2 =

$$\begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

```
R3 = OBJ_P.get_R3
```

R3 =

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi) & \sin(\psi) \\ 0 & -\sin(\psi) & \cos(\psi) \end{pmatrix}$$

Calculate the Direction Cosine Matrix ${}^{B}R_{G}$

Recall we earlier said: ${}^{B}v = {}^{B}R_{G} * {}^{G}v$

```
bRg = R3 * R2 * R1
```

bRg =

$$\begin{pmatrix} \cos(\varphi)\cos(\theta) & \cos(\theta)\sin(\varphi) & -\sin(\theta) \\ \cos(\varphi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\varphi) & \cos(\varphi)\cos(\psi) + \sin(\varphi)\sin(\psi)\sin(\theta) & \cos(\theta)\sin(\psi) \\ \sin(\varphi)\sin(\psi) + \cos(\varphi)\cos(\psi)\sin(\theta) & \cos(\psi)\sin(\varphi)\sin(\theta) - \cos(\varphi)\sin(\psi) & \cos(\psi)\cos(\theta) \end{pmatrix}$$

As a "short distraction" .... it's nice to know I can automatically convert this into a MATLAB function.
NOTE: we're specifying the order of the input variables for the function that gets generated.

```
matlabFunction(bRg,'File','bh_autogen_bRg','Optimize',false, 'Vars', {'phi','theta', 'psi'});

% look at the first 6 lines of this autogenerated file
```

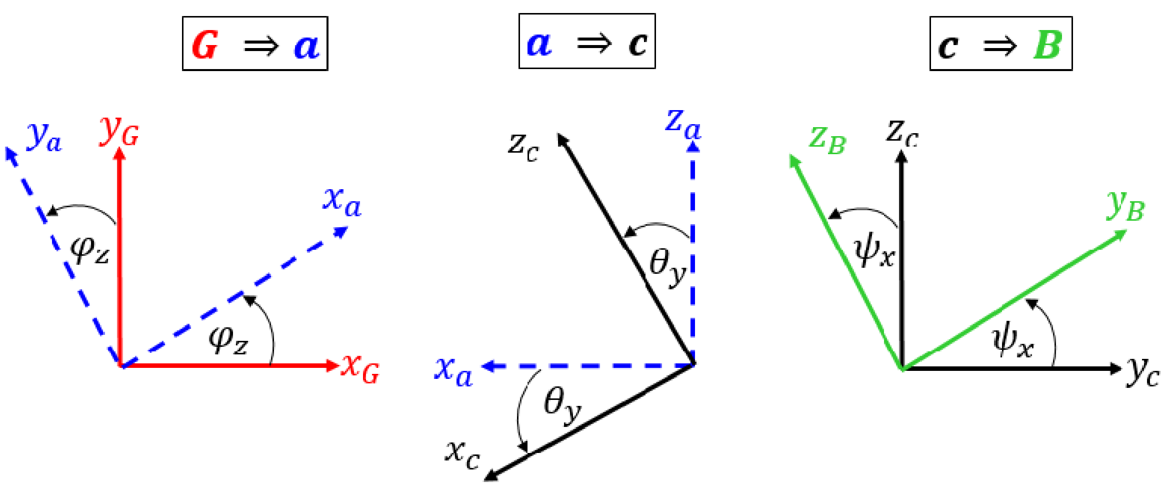
```
dbtype('bh_autogen_bRg', '1:6')
```

```
1       function bRg = bh_autogen_bRg(phi,theta,psi)
2       %BH_AUTOGEN_BRG
3       %    BRG = BH_AUTOGEN_BRG(PHI,THETA,PSI)
4
5       %    This function was generated by the Symbolic Math Toolbox version 7.0.
6       %    06-Jun-2016 08:05:58
```

Explore EULER rates

As we apply these local frame rotations, we can represent the angular rates of the rotating frames in the LOCAL frame co-ordinates. These local frame co-ordinates can then be converted into co-ordinates expressed in the final B frame. For example, during each of the local axes rotations we can think of there being a START frame and an END frame:

Rotation	**START frame**	**End frame**	**Angular rate vector Associated with Rotation**
R1Z(φ)	G-frame	a-frame	$\begin{pmatrix} 0 \\ 0 \\ \varphi\_DOT \end{pmatrix}$ in G
R2Y(θ)	a-frame	c-frame	$\begin{pmatrix} 0 \\ \theta\_DOT \\ 0 \end{pmatrix}$ in a
R3X(ψ)	c-frame	B-frame (the body frame)	$\begin{pmatrix} \psi\_DOT \\ 0 \\ 0 \end{pmatrix}$ in c



We can express each of the local frame angular velocities into their corresponding components in the final B frame - and we'll use PASSIVE rotation matrices to do this:

```
syms phi_dot  theta_dot  psi_dot

aRg = R1;
cRa = R2;
```

```
bRc = R3;

wb_part_1 = bRc * cRa * aRg * [0;0;phi_dot]   % convert local G into B
```

wb_part_1 =

$$\begin{pmatrix} -\varphi_{\mathrm{dot}}\sin(\theta) \\ \varphi_{\mathrm{dot}}\cos(\theta)\sin(\psi) \\ \varphi_{\mathrm{dot}}\cos(\psi)\cos(\theta) \end{pmatrix}$$

```
wb_part_2 = bRc * cRa *       [0;theta_dot;0] % convert local a into B
```

wb_part_2 =

$$\begin{pmatrix} 0 \\ \theta_{\mathrm{dot}}\cos(\psi) \\ -\theta_{\mathrm{dot}}\sin(\psi) \end{pmatrix}$$

```
wb_part_3 = bRc *             [psi_dot;0;0]   % convert local c into B
```

wb_part_3 =

$$\begin{pmatrix} \psi_{\mathrm{dot}} \\ 0 \\ 0 \end{pmatrix}$$

The total angular velocity expressed in the BODY B frame is therefore

We can now construct the total angular velocity vector expressed in components of the final B frame.

$$ {}_G^B\omega_b \equiv \omega_b = f(\phi_{dot}, \theta_{dot}, \psi_{dot})$$

```
wb = wb_part_1 + wb_part_2 + wb_part_3
```

wb =

$$\begin{pmatrix} \psi_{\mathrm{dot}} - \varphi_{\mathrm{dot}}\sin(\theta) \\ \theta_{\mathrm{dot}}\cos(\psi) + \varphi_{\mathrm{dot}}\cos(\theta)\sin(\psi) \\ \varphi_{\mathrm{dot}}\cos(\psi)\cos(\theta) - \theta_{\mathrm{dot}}\sin(\psi) \end{pmatrix}$$

We can write the angular velocity vector $\boldsymbol{\omega_b}$ as a MATRIX equation

Let's say that: $\omega_b = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

We can write a matrix equation of the form **A.x = b** that describes the relationship between the body rates ω_b and the Euler rates:

$$A \times x = b$$

$$A \times \begin{pmatrix} \phi_{dot} \\ \theta_{dot} \\ \psi_{dot} \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \equiv \omega_b$$

```
syms p q r

    x = [phi_dot, theta_dot, psi_dot].'
```

x =

$$\begin{pmatrix} \varphi_{\mathrm{dot}} \\ \theta_{\mathrm{dot}} \\ \psi_{\mathrm{dot}} \end{pmatrix}$$

```
[A,b] = equationsToMatrix(  wb(1)==p, ...
                            wb(2)==q, ...
                            wb(3)==r, ...
                            x)
```

A =

$$\begin{pmatrix} -\sin(\theta) & 0 & 1 \\ \cos(\theta)\sin(\psi) & \cos(\psi) & 0 \\ \cos(\psi)\cos(\theta) & -\sin(\psi) & 0 \end{pmatrix}$$

b =

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

ATTENTION: The SINGULARITY between BODY rates and EULER rates

From the Matrix equation computed above there is actually an angle that causes the determinant of **A** to be ZERO, and hence prevents us from solving for the Euler rates (at that angle) iff we know the body rates ω_b. The angle that causes this problem is the rotation about the local Y axis, ie: the angle θ.

Specifically it is when $\theta = 90^o$. We can see this by first computing the determinant of *A* .

```
det_A = simplify( det(A) )
```

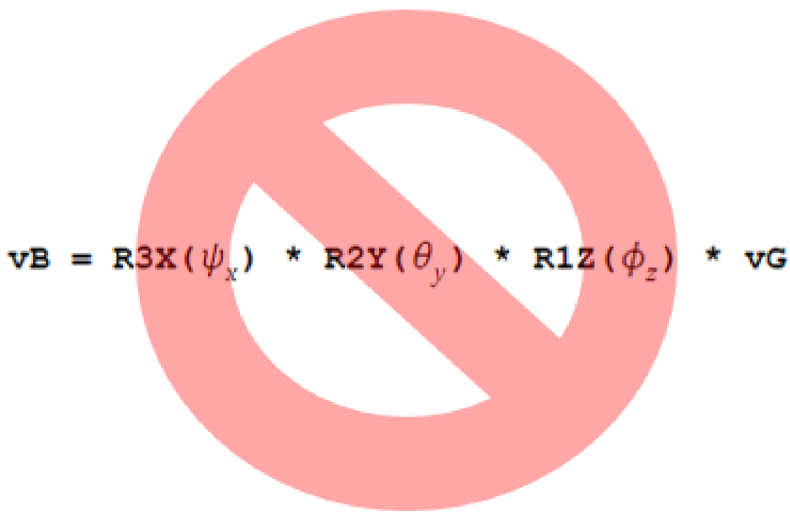
det_A $=-\cos(\theta)$

And then solving for its roots.

```
solve( det_A ==0 )
```

ans $=\frac{\pi}{2}$

So this tells us that as soon as our vehicle has a pitch angle of 90 degrees, that our chosen Euler angle sequence simply canNOT be used to convert body rates ω_b into Euler rates. So? So, if you think your vehicle will pitch by 90 degrees ... ***AND you're wanting to calculate EULER rates from body rates*** .... then you'll need to consider an alternate form of describing your vehicle's pose (eg: quaternions, or integrating directly the DCM)



Let's compute Euler rates from our body rates $\boldsymbol{\omega_b}$

Assuming our vehicle does NOT have a pitch angle of 90 degrees, then we can use the results of the previous section to calculate the Euler rates from our body rates ω_b.

$$Euler_{rates} \equiv \begin{pmatrix} \phi_{dot} \\ \theta_{dot} \\ \psi_{dot} \end{pmatrix} = A^{-1} * \omega_b \text{ where } \omega_b = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

```
euler_rates = inv(A) * [p; q; r];
euler_rates = simplify(euler_rates)
```

euler_rates =

$$\begin{pmatrix} \frac{r\cos(\psi)+q\sin(\psi)}{\cos(\theta)} \\ q\cos(\psi) - r\sin(\psi) \\ \frac{p\cos(\theta)+r\cos(\psi)\sin(\theta)+q\sin(\psi)\sin(\theta)}{\cos(\theta)} \end{pmatrix}$$

We can write the Euler rates vector as a MATRIX equation

Similarly to what we did earlier we can write a matrix equation that describes the relationship between the body rates ω_b and the Euler rates:

$$K \times \omega_b = Euler_{rates}$$

$$K \times \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \phi_{dot} \\ \theta_{dot} \\ \psi_{dot} \end{pmatrix}$$

$$K \times x = b$$

```
x = [p,q,r].'
```

```
x =
```

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

```
[K,b] = equationsToMatrix(  euler_rates(1)==phi_dot,   ...
                            euler_rates(2)==theta_dot, ...
                            euler_rates(3)==psi_dot,   ...
                            x)
```

```
K =
```

$$\begin{pmatrix} 0 & \frac{\sin(\psi)}{\cos(\theta)} & \frac{\cos(\psi)}{\cos(\theta)} \\ 0 & \cos(\psi) & -\sin(\psi) \\ 1 & \frac{\sin(\psi)\sin(\theta)}{\cos(\theta)} & \frac{\cos(\psi)\sin(\theta)}{\cos(\theta)} \end{pmatrix}$$

```
b =
```

$$\begin{pmatrix} \varphi_{\mathrm{dot}} \\ \theta_{\mathrm{dot}} \\ \psi_{\mathrm{dot}} \end{pmatrix}$$