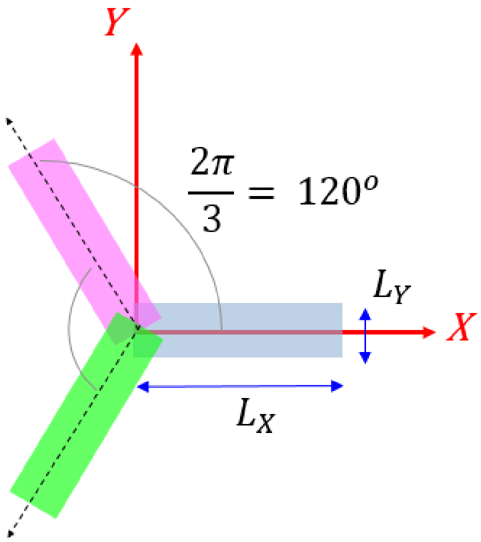
Inertia properties of a 3 blade propeller

What we're going to do:

In this FAQ, we're going to explore the inertia properties of a 3 bladed propeller.



WHY are we doing this?

- We get to practice the calculation of Inertia matrices for "rotated" bodies, eg: parallel axis theorem and PASSIVE rotation matrices.
- The 3 bladed propeller has some inertia matrix properties that will blow your mind !!

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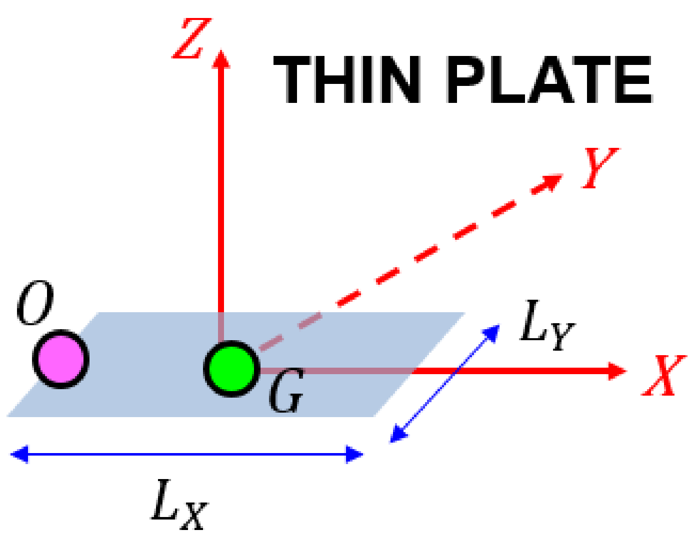
Consider a thin Rectangular plate:

Before we start looking at the 3 bladed propeller, let's quickly review the inertia matrix of a thin rectangular plate. We're doing this because we'll represent a propeller blade as a thin rectangular plate.



In this figure we have a G-frame attached to the Centre of mass of the plate, Let's calculate the inertia of the plate about a parallel frame that is attached at point 0.

```
syms Lx Ly m

% create an instance of a thin rectangular plate class
TRP_OBJ = inertia_thin_rect_plate_CLS(Lx, Ly, m);

% look at the Inertia matrix for the G-frame
TRP_OBJ.get_I()
```

ans =

$$\begin{pmatrix} \frac{Ly^2 m}{12} & 0 & 0 \\ 0 & \frac{Lx^2 m}{12} & 0 \\ 0 & 0 & \frac{m\,(Lx^2+Ly^2)}{12} \end{pmatrix}$$

Now let's apply the Parallel axis theorm to compute the Inertia about the O-frame:

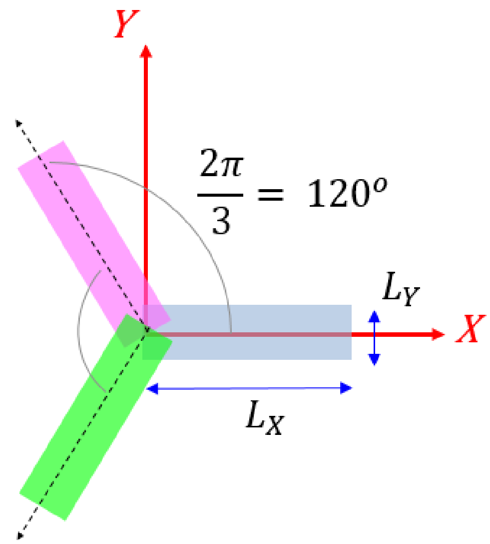
```
% the inertia relative to the G-frame
gI  = TRP_OBJ.get_I();

% define the position of G relative to 0
r_col = [Lx/2, 0, 0].';

% create an instance of the inertia_parallel_local_to_desired_CLS class
OBJ = inertia_parallel_local_to_desired_CLS(r_col, gI, m);

% compute the INERTIA relative to the 0-frame
I_LOCAL_blade = OBJ.calc_I_GLOB()
```

I_LOCAL_blade =

$$\begin{pmatrix} \frac{Ly^2 m}{12} & 0 & 0 \\ 0 & \frac{Lx^2 m}{3} & 0 \\ 0 & 0 & \frac{Lx^2 m}{4} + \frac{m\,(Lx^2+Ly^2)}{12} \end{pmatrix}$$

Now back to the main problem:

Recall what our main problem is. We want to compute the inertia of the 3-bladed propeller relative to the XYZ frame shown below:



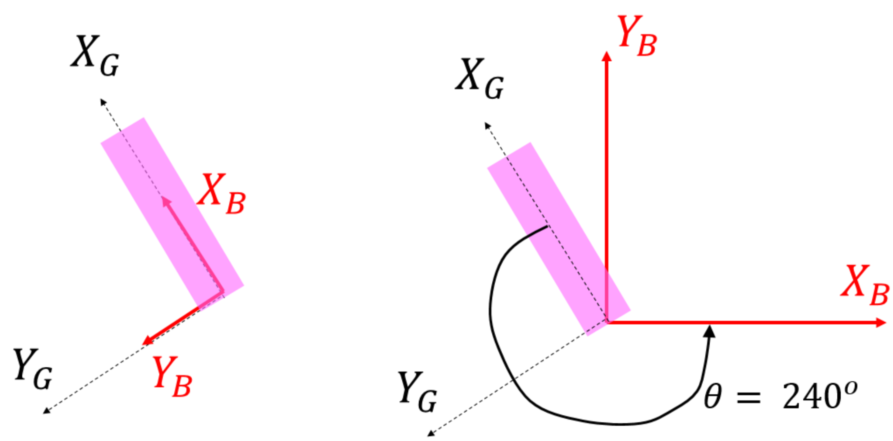
So we can now define I_{BLUE} :

OK, so from our review of the thin rectangular plate AND the application of the paralle axis theorm, we now have the inertia matrix for the BLUE blade.

```
% This is the inertia for our BLUE blade
I_BLUE_blade = I_LOCAL_blade;
```

So how do we compute I_{PINK} and I_{GREEN} :

As a start let's consider the PINK blade.



In the above diagram we have 2 frames: the *G-frame* and the *B-frame*. Initially both of the frames are co-incident. We then rotate the *B-frame* by an angle θ to it's new position.

What we want to do now, is to calculate the inertia matrix of the PINK blade relative to the new position of the *B-frame* shown in the right hand figure, *ie:* ${}^{B}I$. What we know already is the inertia of the pink blade relative to it's local *G-frame, ie:* ${}^{G}I$.

So how do we do calculate ${}^{B}I$? Well the answer starts with our equations for angular momentum. *Consider the following*:

$$ {}^{B}\omega = {}^{B}R_{G} \times {}^{G}\omega \qquad \Longrightarrow \qquad {}^{G}\omega = {}^{B}R_{G}^{T} \times {}^{B}\omega $$

$$ {}^{B}L = {}^{B}R_{G} \times {}^{G}L \qquad \Longrightarrow \qquad {}^{G}L = {}^{B}R_{G}^{T} \times {}^{B}L $$

Now let's focus on the angular momentum described in the G-frame

$$ {}^{G}L = {}^{G}I \times {}^{G}\omega $$

$$ ({}^{B}R_{G}^{T} \times {}^{B}L) = {}^{G}I \times ({}^{B}R_{G}^{T} \times {}^{B}\omega) $$

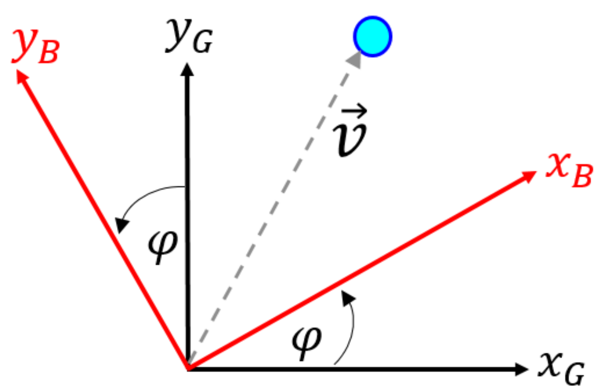
$$ {}^{B}L = ({}^{B}R_{G} \times {}^{G}I \times {}^{B}R_{G}^{T}) \times {}^{B}\omega $$

$$ {}^{B}L = {}^{B}I \times {}^{B}\omega \qquad where \qquad {}^{B}I = ({}^{B}R_{G} \times {}^{G}I \times {}^{B}R_{G}^{T}) $$

The **BIG result** here is this one: $\mathbf{{}^{B}I = ({}^{B}R_{G} \times {}^{G}I \times {}^{B}R_{G}^{T})}$. At the heart of this derivation is a PASSIVE rotation matrix ${}^{B}R_{G}$. This rotation matrix allows us to compute the components of a vector in the B-frame, when we already know the components of the same vector in the G-frame, ie:

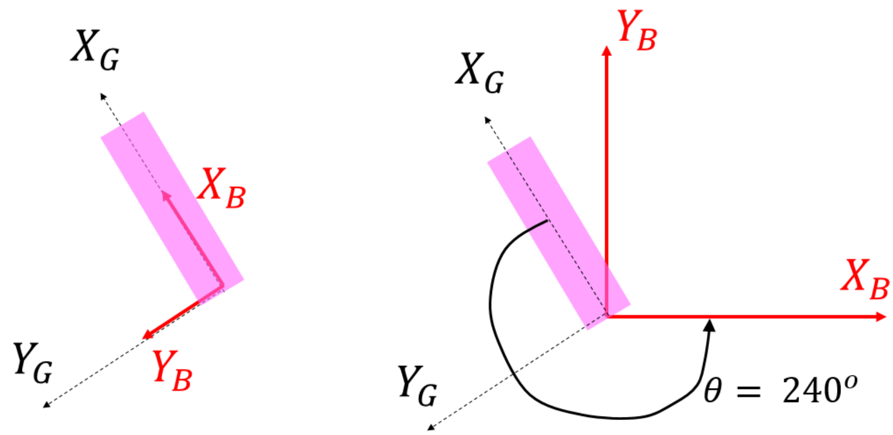
$$ \overrightarrow{{}^{B}v} = {}^{B}R_{G} \times \overrightarrow{{}^{G}v} $$

"think" that the B-frame has rotated relative to the G-frame, ie:



Now let's consider the PINK blade:

As discussed in the previous section, we have a *G-frame* and a *B-frame*. Initially the B and G frames are coincident. The B-frame is then rotated by an angle θ around the Z-axis as shown in the figure below. In our case we have $\theta = 240^{o}$ (the angle is positive because of our right hand rule)




```
% create a PASSIVE rotation object
syms theta
pasR_OBJ = bh_rot_passive_G2B_CLS({'D1Z'}, [ theta ], 'SYM');

% extract the PASIVE rotation matrix bRg
bRg = pasR_OBJ.get_R1
```

bRg =

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
% In our case we have theta = 240 degrees (== 240 * (pi/180) radians)
bRg = subs(bRg, theta, 240*pi/180)
```

bRg =

$$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
% now compute bI for the PINK blade
gI             = I_LOCAL_blade;
I_PINK_blade   = bRg * gI * bRg.'
```

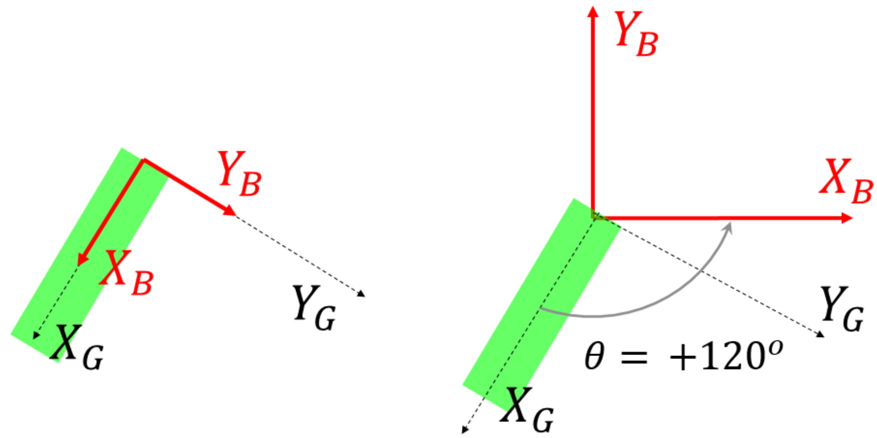
I_PINK_blade =

$$\begin{pmatrix} \frac{m\,\mathrm{Lx}^2}{4} + \frac{m\,\mathrm{Ly}^2}{48} & \sigma_1 & 0 \\ \sigma_1 & \frac{m\,\mathrm{Lx}^2}{12} + \frac{m\,\mathrm{Ly}^2}{16} & 0 \\ 0 & 0 & \frac{\mathrm{Lx}^2\,m}{4} + \frac{m\,(\mathrm{Lx}^2+\mathrm{Ly}^2)}{12} \end{pmatrix}$$

where

$$\sigma_1 = \frac{\sqrt{3}\,\mathrm{Lx}^2\,m}{12} - \frac{\sqrt{3}\,\mathrm{Ly}^2\,m}{48}$$

Now let's consider the GREEN blade:



To calculate the inertia of the GREEN blade relative to the new B-frame, we apply the same analysis as we did with the pink blade.

```
% extract the PASIVE rotation matrix bRg
bRg = pasR_OBJ.get_R1
```

bRg =

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
% In our case we have theta = 120 degrees (== 120*pi/180 radians)
bRg = subs(bRg, theta, 120*pi/180)
```

bRg =

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
% % now compute bI for the GREEN blade
gI            = I_LOCAL_blade;
I_GREEN_blade  = bRg * gI * bRg.'
```

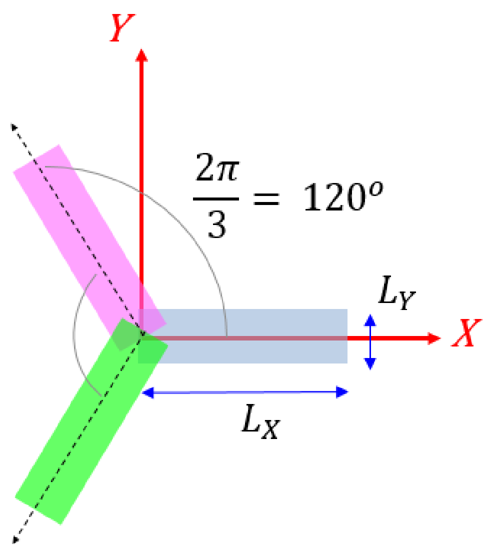
I_GREEN_blade =

$$\begin{pmatrix} \frac{m\,\mathrm{Lx}^2}{4}+\frac{m\,\mathrm{Ly}^2}{48} & \sigma_1 & 0 \\ \sigma_1 & \frac{m\,\mathrm{Lx}^2}{12}+\frac{m\,\mathrm{Ly}^2}{16} & 0 \\ 0 & 0 & \frac{\mathrm{Lx}^2\,m}{4}+\frac{m\,(\mathrm{Lx}^2+\mathrm{Ly}^2)}{12} \end{pmatrix}$$

where

$$\sigma_1 = \frac{\sqrt{3}\,\mathrm{Ly}^2\,m}{48} - \frac{\sqrt{3}\,\mathrm{Lx}^2\,m}{12}$$

Now assemble the inertias of all three blades:




```
I_sys_config_1 = I_BLUE_blade + I_PINK_blade + I_GREEN_blade;
simplify(I_sys_config_1)
```

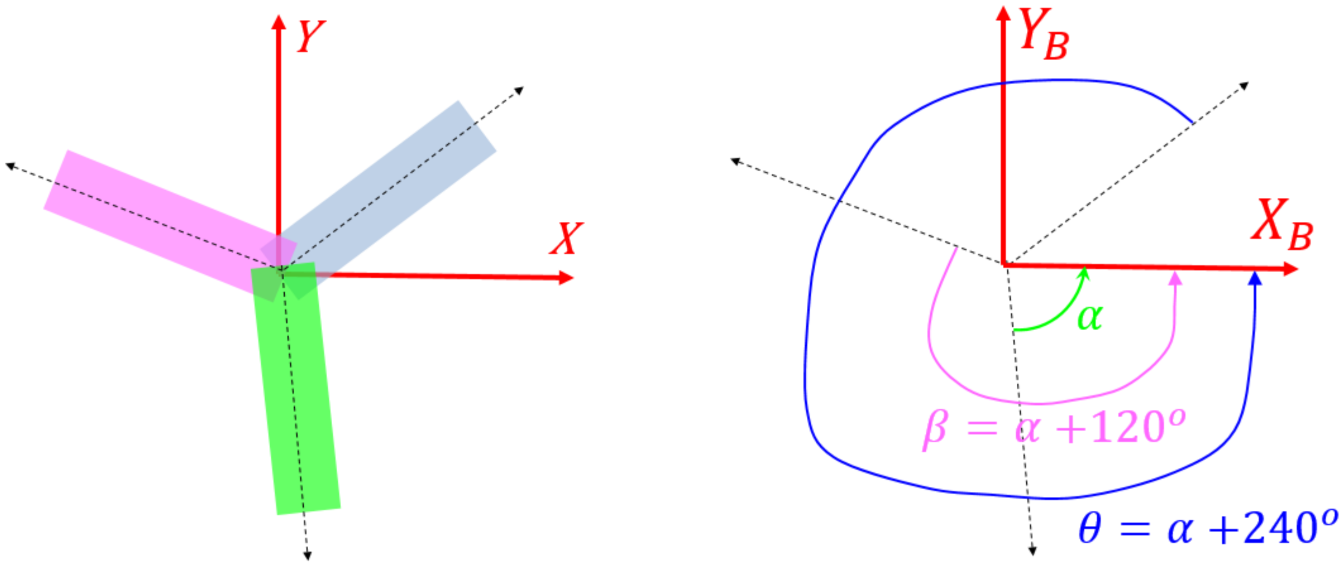
ans =

$$\begin{pmatrix} \frac{m\,(4\,\mathrm{Lx}^2+\mathrm{Ly}^2)}{8} & 0 & 0 \\ 0 & \frac{m\,(4\,\mathrm{Lx}^2+\mathrm{Ly}^2)}{8} & 0 \\ 0 & 0 & \frac{m\,(4\,\mathrm{Lx}^2+\mathrm{Ly}^2)}{4} \end{pmatrix}$$

Note from the above system INERTIA matrix that our product of inertia terms (eg: I_{XY} , I_{YZ} , etc) are all ZERO. And note also that the I_{XX} and I_{YY} moment of inertia terms are identical. **This is cool !** .... But is it "lucky" cool or is there something deeper here that we need to explore ?

Consider an arbitrarily orientated propeller:

Consider the following arbitrarily orientated propeller system:




```
% create a PASSIVE rotation object
syms alpha
GREEN_OBJ = bh_rot_passive_G2B_CLS({'D1Z'}, [   alpha ],               'SYM');
PINK_OBJ  = bh_rot_passive_G2B_CLS({'D1Z'}, [ (alpha + 120*pi/180) ], 'SYM');
BLUE_OBJ  = bh_rot_passive_G2B_CLS({'D1Z'}, [ (alpha + 240*pi/180) ], 'SYM');

% Have a look at each of the PASSIVE rotation matrices bRg
GREEN_bRg = GREEN_OBJ.get_R1
```

GREEN_bRg =

$$\begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
PINK_bRg  = PINK_OBJ.get_R1
```

PINK_bRg =

$$\begin{pmatrix} \cos\left(\alpha + \frac{2\pi}{3}\right) & \sin\left(\alpha + \frac{2\pi}{3}\right) & 0 \\ -\sin\left(\alpha + \frac{2\pi}{3}\right) & \cos\left(\alpha + \frac{2\pi}{3}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
BLUE_bRg  = BLUE_OBJ.get_R1
```

BLUE_bRg =

$$\begin{pmatrix} \cos\left(\alpha + \frac{4\pi}{3}\right) & \sin\left(\alpha + \frac{4\pi}{3}\right) & 0 \\ -\sin\left(\alpha + \frac{4\pi}{3}\right) & \cos\left(\alpha + \frac{4\pi}{3}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
% define a function for computing the XY frame inertia for each blade
I_fh = @(bRg, Ig)(bRg * Ig * bRg.');

% calculate the inertias relative to the X,Y frame
gI             = I_LOCAL_blade;
I_GREEN_blade = I_fh(GREEN_bRg,  gI);
I_PINK_blade  = I_fh( PINK_bRg,  gI);
I_BLUE_blade  = I_fh( BLUE_bRg,  gI);

% combine for the SYSTEM inertia matrix
I_sys_config_arb = I_GREEN_blade + I_PINK_blade + I_BLUE_blade;
simplify(I_sys_config_arb)
```

```
ans =
```

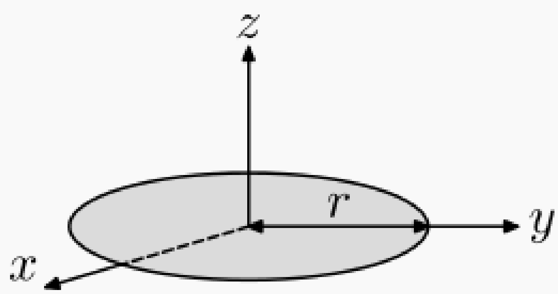
$$\begin{pmatrix} \frac{m\ (4\,\mathrm{Lx}^2+\mathrm{Ly}^2)}{8} & 0 & 0 \\ 0 & \frac{m\ (4\,\mathrm{Lx}^2+\mathrm{Ly}^2)}{8} & 0 \\ 0 & 0 & \frac{m\ (4\,\mathrm{Lx}^2+\mathrm{Ly}^2)}{4} \end{pmatrix}$$

Note from the above system INERTIA matrix, that even when the propeller is placed in an arbitrary pose, the product of inertia terms (eg: I_{XY} , I_{YZ} , etc) are still all ZERO and our 2 moment of inertia terms I_{XX} and I_{YY} are still identical (ie: $I_{XX} = I_{YY}$). Note also that the pose angle α does NOT appear in the Inertia matrix ! - so regardless of the in plane orientation of the propeller, the INERTIA matrix is always the same !

So? - So the 3 bladed propeller has INERTIA properties that are ***similar*** to a thin circular disc.

This is truly an amazing result !

*FYI: Here are the inertia values fro a circular disk (see **REF**)*



$$I_z = \frac{mr^2}{2}$$

$$I_x = I_y = \frac{mr^2}{4}$$