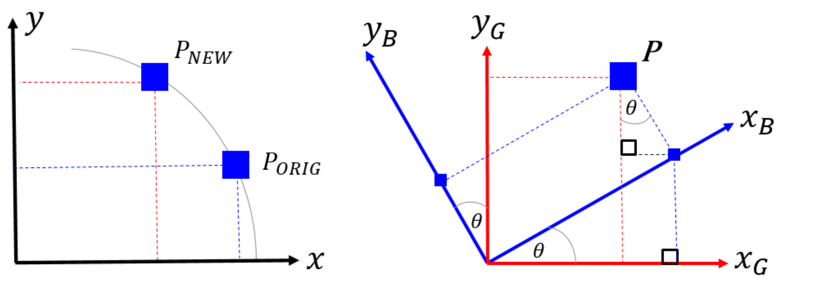
Explore **ACTIVE** rotations applied to a BODY-FIXED frame

In this tutorial, we're going to explore the concept of **ACTIVE** rotation matrices.



$$\begin{bmatrix} x_P \\ y_P \end{bmatrix}_G = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} * \begin{bmatrix} x_P \\ y_P \end{bmatrix}_B$$

$${}^{G}P = {}^{G}R_{B} * {}^{B}P$$

Why are we doing this ?

- Rotation matrices are used heavily in Mechanical, Robotic and Aeronautical engineering applications.
- Often students can get confused when they read the term "Rotation matrix". In many/most cases, this confusion can be reduced by emphasizing a rotation matrix as being either PASSIVE or ACTIVE.

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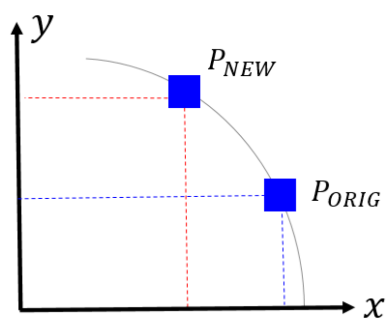
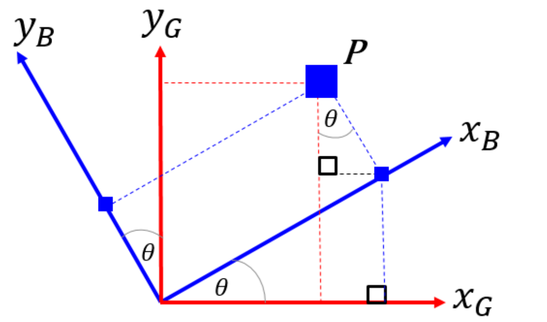
Introduction:

Consider the scenario where we have an original data point P_{ORIG} and we want to rotate this data point to a new location called P_{NEW} . This scenario is shown below. We can think of this task in the following way:

- Imagine that we start with P_{ORIG} and that this point is fixed ("glued") to a co-ordinate frame called the **B-frame**.
- We know the *(x,y)* co-ordinates of the point in this **B-fame** and refer to this as ${}^{B}P$
- We then rotate the **B-frame** relative to a fixed frame called the **G-frame**. Note that because point P is "glued" to the **B-frame**, the co-ordinates ${}^{B}P$ do not change while the **B-frame** is rotating.

We now want to now what the final co-ordinate of the point P is relative to the fixed **G-frame,** ie: what is ${}^{G}P$? This is also shown below:

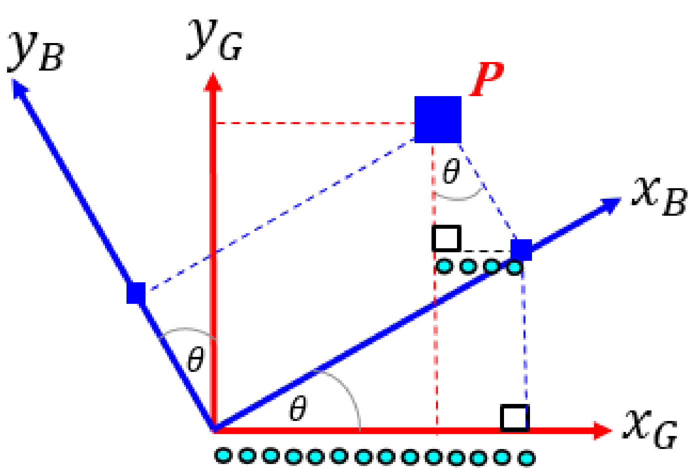


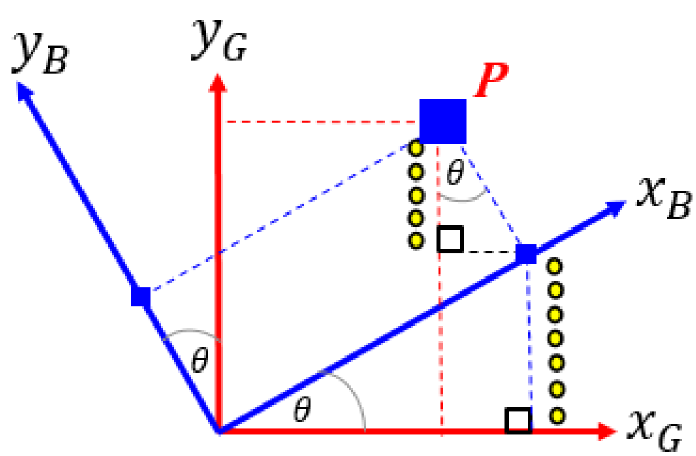


$$\begin{bmatrix} x_P \\ y_P \end{bmatrix}_G = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} * \begin{bmatrix} x_P \\ y_P \end{bmatrix}_B$$

$$^{G}P = {}^{G}R_B * {}^{B}P$$

An **ACTIVE** rotation matrix ${}^{G}R_B$, allows us to calculate the position of the new point relative to the G-frame, ie: ${}^{G}P$. An example of a matrix equation that defines this **ACTIVE** rotation is defined below:





$$\begin{bmatrix} x_P \\ y_P \end{bmatrix}_G = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} * \begin{bmatrix} x_P \\ y_P \end{bmatrix}_B$$

A concrete example - part 1:

Consider the specific case of ${}^{B}P = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and a B-frame rotated by 45 degrees relative to Z axis of the G-frame:

```
bP    = [1,0,0]';
alpha = 45*pi/180;

gRb   = [   cos(alpha), -sin(alpha),   0;
            sin(alpha),  cos(alpha),   0;
                     0,           0,   1];
```

```
gP = gRb * bP
```

```
gP =
         0.707106781186548
         0.707106781186547
                         0
```

A concrete example - part 2:

We can implement the formula for this ACTIVE rotation matrix ${}^{G}R_{B}$ into a MATLAB class called <**`bh_rot_active_B2G_CLS`**>. This will allow us to reuse the formula over and over again. So repeating the previous example we have:

```
bP    = [1, 0, 0]';
alpha = 45*pi/180;
OBJ_AR = bh_rot_active_B2G_CLS({'D1Z'}, alpha, 'RADIANS');
gRb    = OBJ_AR.get_active_R1();

gP = gRb * bP
```

```
gP =
         0.707106781186548
         0.707106781186547
                         0
```

Recall our discussion on PASSIVE rotations

There is a special relationship between PASSIVE and ACTIVE rotation matrices, so let's first review what we know about PASSIVE rotation matrices. Say we have a fixed G-frame. We start by having our B-frame co-incident with G, and then we start to rotate the B-frame. Specifically, we're going to apply 3 LOCAL axes rotations which will result in a newly orientated frame called the B-frame. Assume that we apply these 3 successive rotations in the following order:

1. R1Z occurs 1st about the LOCAL **Z** body axis (ϕ), aka **YAW**
2. R2Y occurs 2nd about the LOCAL **Y** body axis (θ), aka **PITCH**
3. R3X occurs 3rd about the LOCAL **X** body axis (ψ), aka **ROLL**

We can express a vector defined in the G-frame to it's corresponding description in the B-frame, using a sequence of **PASSIVE** rotation matrices, ie:

$$\mathbf{{}^{B}v = R3X(\psi_x) \times R2Y(\theta_y) \times R1Z(\phi_z) \times {}^{G}v}$$

OR, in a more compact form as: $\mathbf{{}^{B}v = {}^{B}R_{G} \times {}^{G}v}$ Where ${}^{B}R_{G}$ is the PASSIVE rotation matrix.

Now define what we mean by ACTIVE rotations

Continuing on from the previous section, we can now write:

$${}^{G}\mathbf{v} = R1Z(\phi_z)^{-1} * R2Y(\theta_y)^{-1} * R3X(\psi_x)^{-1} * {}^{B}\mathbf{v}$$

$${}^{G}\mathbf{v} = R1Z(\phi_z)^{T} * R2Y(\theta_y)^{T} * R3X(\psi_x)^{T} * {}^{B}\mathbf{v}$$

$${}^{G}\mathbf{v} = R1Z(-\phi_z) * R2Y(-\theta_y) * R3X(-\psi_x) * {}^{B}\mathbf{v}$$

If we now define the following **ACTIVE** rotation matrices:

1. **a_R1Z(** ϕ_z **)** = $R1Z(\phi_z)^{-1}$ = $R1Z(-\phi_z)$
2. **a_R2Y(** θ_y **)** = $R2Y(\theta_y)^{-1}$ = $R2Y(-\theta_y)$
3. **a_R3X(** ψ_x **)** = $R3X(\psi_x)^{-1}$ = $R3X(-\psi_x)$

Then we can write:

${}^{G}\mathbf{v}$ = **a_R1Z(** ϕ_z **)** * **a_R2Y(** θ_y **)** * **a_R3(** ψ_x **)** * ${}^{B}\mathbf{v}$

Or in a more compact form: ${}^{G}\mathbf{v} = {}^{G}\mathbf{R}_{\mathbf{B}} \times {}^{B}\mathbf{v}$ Where ${}^{G}R_{B}$ is the ACTIVE rotation matrix.

It should be clear that : ${}^{G}R_{B} = ({}^{B}R_{G})^{-1} = ({}^{B}R_{G})^{T}$

Let's explore these ACTIVE rotations

Let's create on of those active rotation objects that we used earlier. We'll create an object that implmenets the sequence:

1. R1Z occurs 1st about the LOCAL **Z** body axis (ϕ), aka **YAW**
2. R2Y occurs 2nd about the LOCAL **Y** body axis (θ), aka **PITCH**
3. R3X occurs 3rd about the LOCAL **X** body axis (ψ), aka **ROLL**

```
OBJ_A = bh_rot_active_B2G_CLS({'D1Z', 'D2Y', 'D3X'}, [sym('phi'), sym('theta'), sym('psi')], '
```

```
OBJ_A = 
  bh_rot_active_B2G_CLS with properties:

        ang_units: SYM
    num_rotations: 3
          dir_1st: D1Z
          dir_2nd: D2Y
          dir_3rd: D3X
          ang_1st: [1x1 sym]
          ang_2nd: [1x1 sym]
          ang_3rd: [1x1 sym]
```

The symbolic ACTIVE rotation matrices

```
aR1     = OBJ_A.get_active_R1
```

aR1 =

$$\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
aR2     = OBJ_A.get_active_R2
```

aR2 =

$$\begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

```
aR3     = OBJ_A.get_active_R3
```

aR3 =

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi) & -\sin(\psi) \\ 0 & \sin(\psi) & \cos(\psi) \end{pmatrix}$$

Here are some compound ACTIVE rotation matrices - part 1

```
aR1R2    = aR1*aR2
```

aR1R2 =

$$\begin{pmatrix} \cos(\varphi)\cos(\theta) & -\sin(\varphi) & \cos(\varphi)\sin(\theta) \\ \cos(\theta)\sin(\varphi) & \cos(\varphi) & \sin(\varphi)\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

Note that "aR1 * R2" is the same thing as "get_active_R1R2()":

```
diff_mat = aR1R2 - OBJ_A.get_active_R1R2 % this should be a ZERO matrix
```

diff_mat =

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Here are some compound ACTIVE rotation matrices - part 2

```
aR1R2R3 = aR1*aR2*aR3
```

```
aR1R2R3 =
```

$$\begin{pmatrix} \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\varphi) & \sin(\varphi)\sin(\psi) + \cos(\varphi)\cos(\psi)\sin(\theta) \\ \cos(\theta)\sin(\varphi) & \cos(\varphi)\cos(\psi) + \sin(\varphi)\sin(\psi)\sin(\theta) & \cos(\psi)\sin(\varphi)\sin(\theta) - \cos(\varphi)\sin(\psi) \\ -\sin(\theta) & \cos(\theta)\sin(\psi) & \cos(\psi)\cos(\theta) \end{pmatrix}$$

Note that "aR1 * R2 * aR3" is the same thing as "get_active_R1R2R3()":

```
diff_mat = aR1R2R3 - OBJ_A.get_active_R1R2R3 % this should be a ZERO matrix
```

```
diff_mat =
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Here is the ACTIVE rotation matrix ${}^{G}\boldsymbol{R}_{B}$

Here is the compound ACTIVE rotation matrix:

```
gRb = aR1*aR2*aR3
```

```
gRb =
```

$$\begin{pmatrix} \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\varphi) & \sin(\varphi)\sin(\psi) + \cos(\varphi)\cos(\psi)\sin(\theta) \\ \cos(\theta)\sin(\varphi) & \cos(\varphi)\cos(\psi) + \sin(\varphi)\sin(\psi)\sin(\theta) & \cos(\psi)\sin(\varphi)\sin(\theta) - \cos(\varphi)\sin(\psi) \\ -\sin(\theta) & \cos(\theta)\sin(\psi) & \cos(\psi)\cos(\theta) \end{pmatrix}$$

Recall the PASSIVE rotation matrix ${}^{B}\boldsymbol{R}_{G}$

Note how the inverse of the **ACTIVE** ${}^{G}R_{B}$ is just the **PASSIVE** ${}^{B}R_{G}$ which we computed during our discussion on PASSIVE rotations

```
bRg = inv(gRb);
simplify(bRg)
```

```
ans =
```

$$\begin{pmatrix} \cos(\varphi)\cos(\theta) & \cos(\theta)\sin(\varphi) & -\sin(\theta) \\ \cos(\varphi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\varphi) & \cos(\varphi)\cos(\psi) + \sin(\varphi)\sin(\psi)\sin(\theta) & \cos(\theta)\sin(\psi) \\ \sin(\varphi)\sin(\psi) + \cos(\varphi)\cos(\psi)\sin(\theta) & \cos(\psi)\sin(\varphi)\sin(\theta) - \cos(\varphi)\sin(\psi) & \cos(\psi)\cos(\theta) \end{pmatrix}$$

Let's rotate an aeriel vehicle:

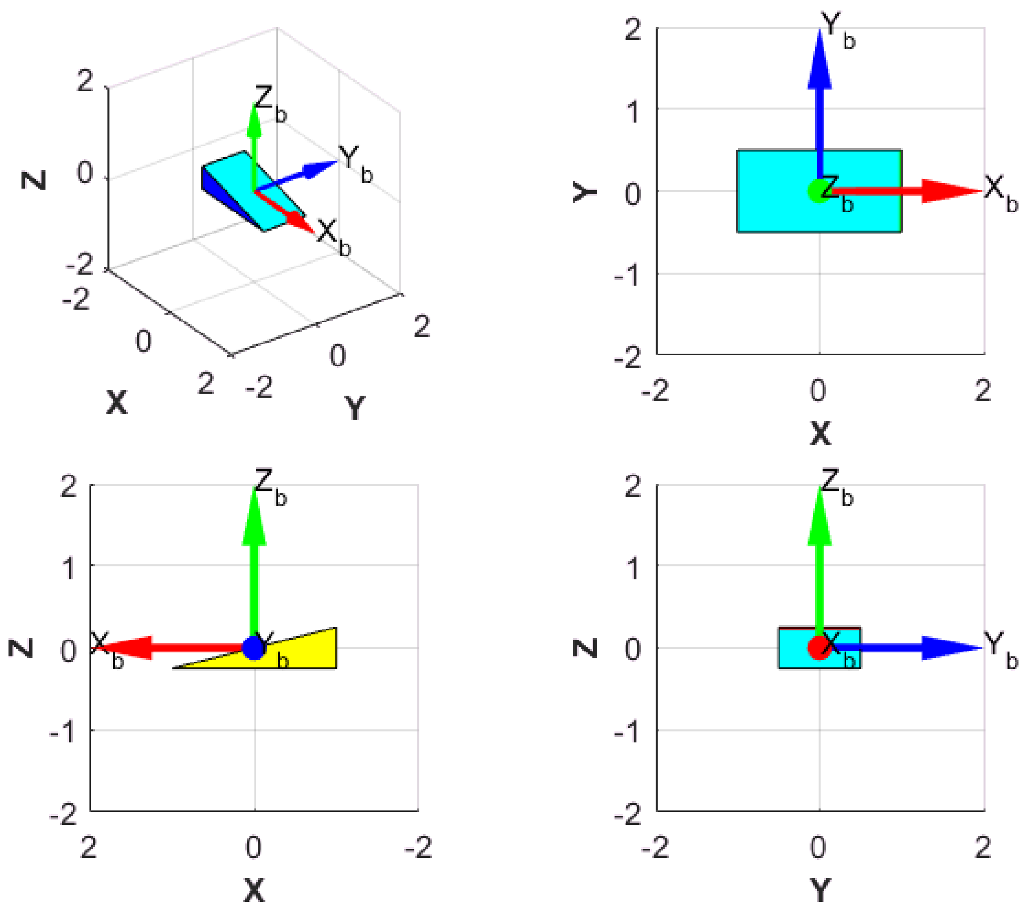
Now let's apply these ACTIVE rotation matrices to a "vehicle":

```
% this will be the "toy" system that we'll rotate in space
veh_OBJ = bh_vehicle_CLS()
```

```
veh_OBJ = 
  bh_vehicle_CLS with properties:

    FaceAlpha: 1
          gRb: [3x3 double]
           XL: [-2 2]
           YL: [-2 2]
           ZL: [-2 2]
      X_b_col: [18x1 double]
      Y_b_col: [18x1 double]
      Z_b_col: [18x1 double]
      X_g_col: [18x1 double]
      Y_g_col: [18x1 double]
      Z_g_col: [18x1 double]
```

Show the vehicle in it's original pose

```
figure();
hax(1) = subplot(2,2,1);  veh_OBJ.plot_3D(hax(1));
hax(2) = subplot(2,2,2);  veh_OBJ.plot_XY(hax(2));
hax(3) = subplot(2,2,3);  veh_OBJ.plot_XZ(hax(3));
hax(4) = subplot(2,2,4);  veh_OBJ.plot_YZ(hax(4));
```



Define the ACTIVE rotation sequence and angles

We'd like to subject the vehicle to a series of rotations applied to a body fixed co-ordinate frame attached to the vehicle. Assume that we apply these 3 successive rotations in the following order:

1. R1Z occurs 1st about the LOCAL **Z** body axis (ϕ), aka **YAW**
2. R2Y occurs 2nd about the LOCAL **Y** body axis (θ), aka **PITCH**
3. R3X occurs 3rd about the LOCAL **X** body axis (ψ), aka **ROLL**

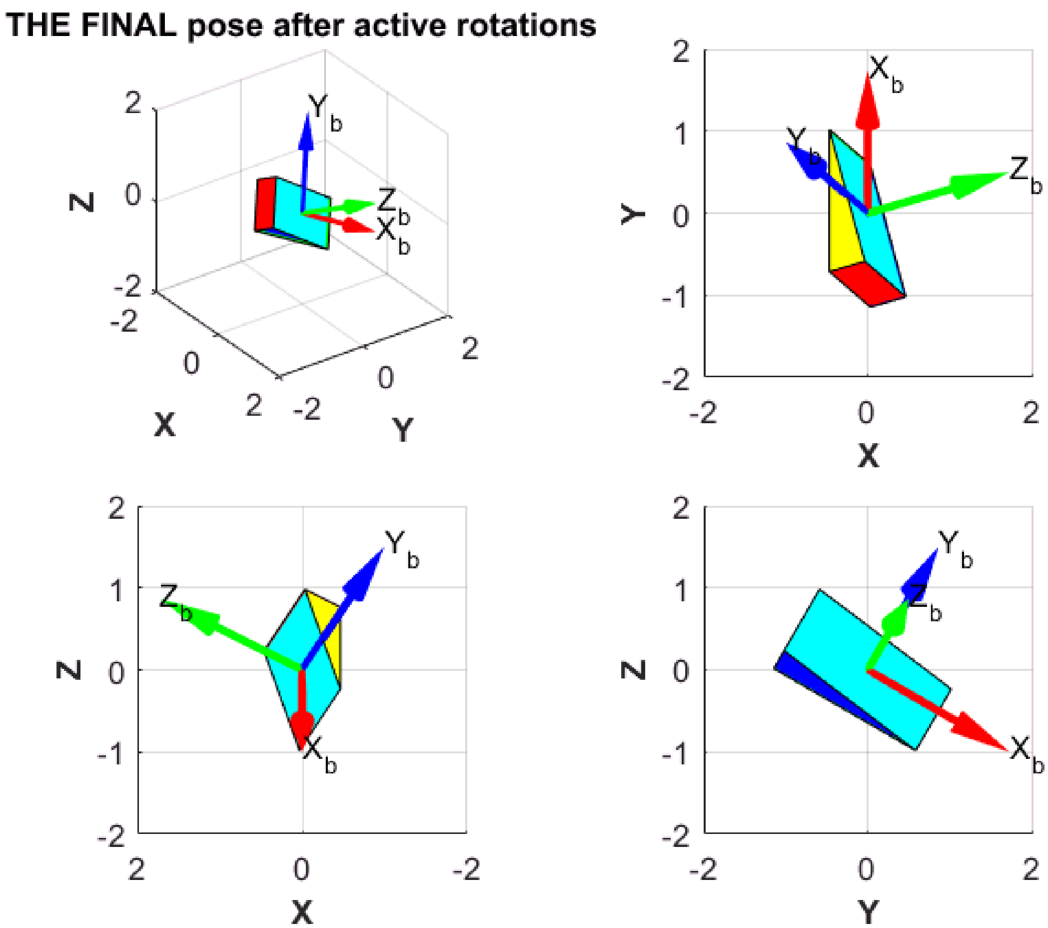
```
degs_yaw  = 90;
degs_pitch= 30;
degs_roll = 60;

arot_OBJ  = bh_rot_active_B2G_CLS({'D1Z','D2Y','D3X'}, ...
                                 [degs_yaw, degs_pitch, degs_roll], ...
                                 'DEGREES')
```

```
arot_OBJ =
  bh_rot_active_B2G_CLS with properties:

        ang_units: DEGREES
    num_rotations: 3
          dir_1st: D1Z
          dir_2nd: D2Y
          dir_3rd: D3X
          ang_1st: 90
          ang_2nd: 30
          ang_3rd: 60
```

Now apply this ACTIVE rotation sequence to the vehicle

```
% get each of the active rotation matrices
aR1 = arot_OBJ.get_active_R1();
aR2 = arot_OBJ.get_active_R2();
aR3 = arot_OBJ.get_active_R3();

% chain them together in the correct ACTIVE order
aR1R2R3 = aR1 * aR2 * aR3;

% get the B frame geometry data of the vehicle
[X,Y,Z] = veh_OBJ.get_B_XYZ();
v_mat   = [ X(:), Y(:), Z(:) ]';  % a 3xN matrix

% now apply the complete ACTIVE rotation matrix to our vehicle data
new_XYZ = aR1R2R3 * v_mat;

% store this new rotated vehicle data
veh_OBJ       =  veh_OBJ.set_G_XYZ(new_XYZ(1,:)', new_XYZ(2,:)', new_XYZ(3,:)');

% store the DCM so that we can draw the body fixed frame arrows
veh_OBJ.gRb = arot_OBJ.get_active_R;

% plot the new rotated vehicle
figure();
hax(1) = subplot(2,2,1);  veh_OBJ.plot_3D(hax(1));
```

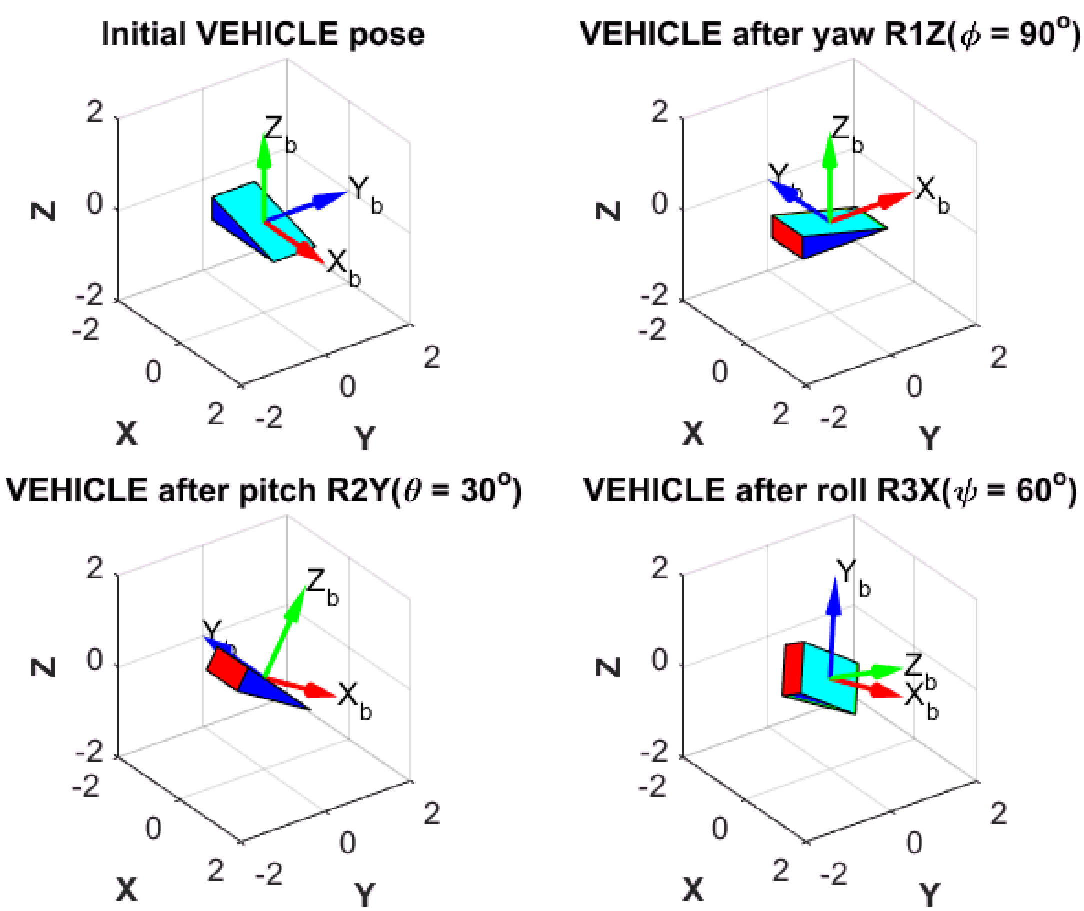
```
hax(2) = subplot(2,2,2);  veh_OBJ.plot_XY(hax(2));
hax(3) = subplot(2,2,3);  veh_OBJ.plot_XZ(hax(3));
hax(4) = subplot(2,2,4);  veh_OBJ.plot_YZ(hax(4));
title(hax(1), 'THE FINAL pose after active rotations')
```



REPEAT what we just did ... BUT let's show the progressive rotations

```
veh_OBJ = bh_vehicle_CLS();
figure();
clear hax

% Here's the vehicle in its ORIGINAL pose
hax(1) = subplot(2,2,1);  veh_OBJ.plot_3D(hax(1));
title(hax(1),'Initial VEHICLE pose')

% apply the 1st active rotation
clear veh_OBJ
veh_OBJ = bh_vehicle_CLS();                % ORIG pose is starting point
V_3xN   =  veh_OBJ.get_B_XYZ_3xN();        % get current vehicle data
new_XYZ = arot_OBJ.apply_active_R1(V_3xN); % apply the rotation
veh_OBJ =  veh_OBJ.set_G_XYZ(new_XYZ(1,:)', new_XYZ(2,:)', new_XYZ(3,:)');
gRb     = arot_OBJ.get_active_R1();        % get and store the DCM
veh_OBJ.gRb = gRb;
% update the vehicle's PLOT
hax(2)  = subplot(2,2,2);  veh_OBJ.plot_3D(hax(2));
str     = sprintf('VEHICLE after yaw R1Z(\\phi = %d^o)',degs_yaw);
title(hax(2),str)
```

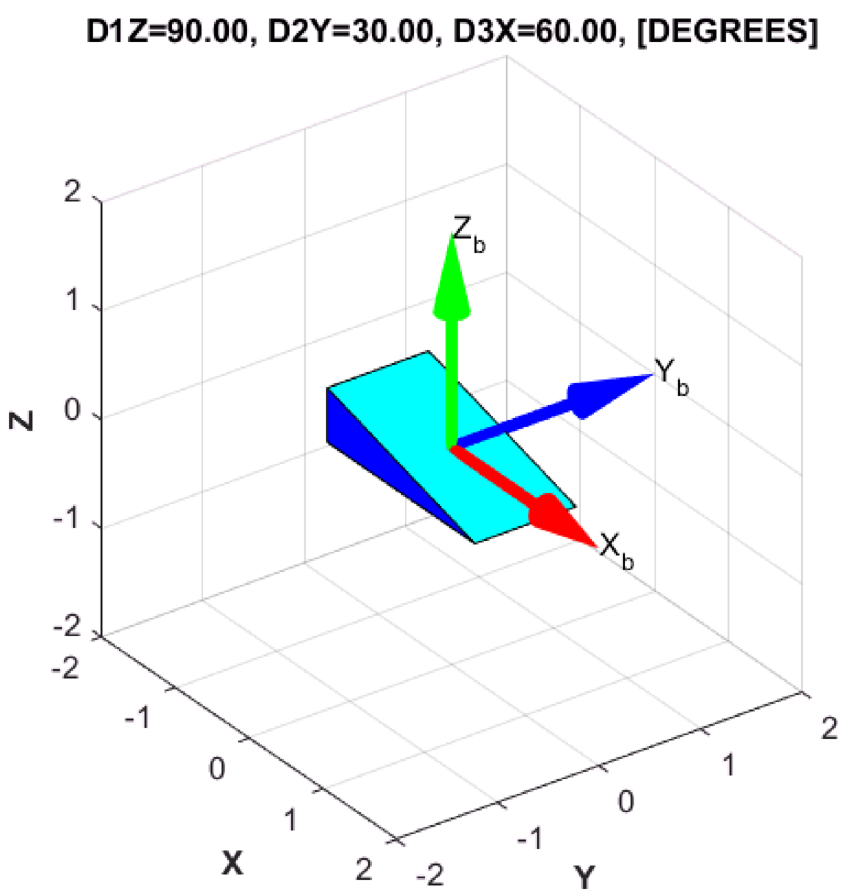
```
% apply the 2nd active multiplication
clear veh_OBJ
veh_OBJ = bh_vehicle_CLS();                    % ORIG pose is starting point
V_3xN   =  veh_OBJ.get_B_XYZ_3xN();            % get current vehicle data
new_XYZ = arot_OBJ.apply_active_R1R2(V_3xN); % apply the rotation
veh_OBJ =  veh_OBJ.set_G_XYZ(new_XYZ(1,:)', new_XYZ(2,:)', new_XYZ(3,:)');
gRb     = arot_OBJ.get_active_R1R2();
veh_OBJ.gRb = gRb;
% update the vehicle's PLOT
hax(3)  = subplot(2,2,3);  veh_OBJ.plot_3D(hax(3));
str     = sprintf('VEHICLE after pitch R2Y(\\theta = %d^o)',degs_pitch);
title(hax(3),str)

% apply the 3rd active multiplication
clear veh_OBJ
veh_OBJ = bh_vehicle_CLS();                      % ORIG pose is starting point
V_3xN   =  veh_OBJ.get_B_XYZ_3xN();              % get current vehicle data
new_XYZ = arot_OBJ.apply_active_R1R2R3(V_3xN); % apply the rotation
veh_OBJ =  veh_OBJ.set_G_XYZ(new_XYZ(1,:)', new_XYZ(2,:)', new_XYZ(3,:)');
gRb     = arot_OBJ.get_active_R1R2R3();
veh_OBJ.gRb = gRb;
% update the vehicle's PLOT
hax(4)  = subplot(2,2,4);   veh_OBJ.plot_3D(hax(4));
str     = sprintf('VEHICLE after roll R3X(\\psi = %d^o)',degs_roll);
title(hax(4),str)
```



Next steps:

Animating what we've just done. If you evaluate the following code in the MATLAB command window, you'll see an animation of our vehicle:

```
% Create an ACTIVE rotation object

degs_yaw   = 90;
degs_pitch= 30;
degs_roll = 60;
arot_OBJ   = bh_rot_active_B2G_CLS({'D1Z','D2Y','D3X'}, ...
                                  [degs_yaw, degs_pitch, degs_roll], ...
                                  'DEGREES');
% create a figure
figure();
hax = axes;
desc_str = arot_OBJ.get_description();
title(hax, desc_str);
```




```
% create an ANIMATION
veh_OBJ = veh_OBJ.rotate_and_animate(arot_OBJ, hax);
```

D1Z=90.00, D2Y=30.00, D3X=60.00, [DEGREES]